Combinatorics of minimal absent words for a sliding window

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Abstract

A string w is called a minimal absent word (MAW) for another string T if w does not occur in T but the proper substrings of w occur in T. For example, let $\Sigma = \{a, b, c\}$ be the alphabet. Then, the set of MAWs for string w = abaab is $\{aaa, aaba, bab, bb, c\}$. In this paper, we study combinatorial properties of MAWs in the sliding window model, namely, how the set of MAWs changes when a sliding window of fixed length d is shifted over the input string T of length n, where $1 \le d < n$. We present tight upper and lower bounds on the maximum number of changes in the set of MAWs for a sliding window over T, both in the cases of general alphabets and binary alphabets. Our bounds improve on the previously known best bounds [Crochemore et al., 2020].

1 Introduction

We say that a string u occurs in another string T if s is a substring of T. A non-empty string w is said to be a minimal absent word (an MAW) for a string T if w does not occur in T but any proper substring of w occurs in T. Note that by definition a string of length 1 (namely a character) which does not occur in T is also an MAW for T. On the other hand, any MAW for T of length at least 2 can be represented as aub, where a and b are single characters and u is a (possibly empty) string, such that both au and ub occur in T. For example, let $\Sigma = \{a, b, c\}$ be the alphabet. Then, the set of MAWs for string w = abaab is $\{aaa, aaba, bab, bb, c\}$.

Applications of (minimal) absent words include phylogeny [6], data compression [11], musical information retrieval [8], and bioinformatics [1, 7].

1.1 Algorithms for finding MAWs for string

Given the afore-mentioned motivations, finding MAWs from a given string has been an important and interesting string algorithmic problem and several nice solutions have been

proposed. The first non-trivial algorithm, which was given by Crochemore et al. [10], finds the set $\mathsf{MAW}(T)$ of all MAWs for a given string T of length n over an alphabet of size σ in $\Theta(\sigma n)$ time with O(n) working space. Since $|\mathsf{MAW}(T)| = O(\sigma n)$ for any string T of length n and $|\mathsf{MAW}(S)| = \Omega(\sigma n)$ for some string S of length n [10], Crochemore et al.'s algorithm [10] runs in optimal time in the worst case. Fujishige et al. [13] improved Crochemore et al.'s algorithm so that $\mathsf{MAW}(T)$ can be computed in output-sensitive $O(n + |\mathsf{MAW}(T)|)$ time with O(n) working space. Both of these algorithms use the directed acyclic word graph (DAWG) [5] of string T as a powerful tool for enumerating all MAWs for T. Belazzougui et al. [4] showed that $\mathsf{MAW}(T)$ can also be computed in $O(n + |\mathsf{MAW}(T)|)$ time, provided that the bidirectional Burrows-Wheeler transform of a given string has already been computed. Barton et al. [2] proposed a practical algorithm to compute $\mathsf{MAW}(T)$ in $\Theta(n\sigma)$ time and working space¹ based on the suffix array [14] of T. A parallel algorithm for computing MAWs has also been proposed [3]. Fici and Gawrychowski [12] extended the notion of MAWs to rooted/unrooted labeled trees and presented efficient algorithms to compute them.

1.2 MAWs for sliding window

This paper follows the recent line of research on MAWs for the sliding window model, which was initiated by Crochemore et al. [9]. In this model, the goal is to compute or analyze $\mathsf{MAW}(T[i..i+d-1])$ for every window T[i..i+d-1] of fixed length $d \geq 1$ that shifts T from left to right with increasing $i=1,\ldots,n-d+1$. For instance, consider

Crochemore et al. [9] presented a suffix-tree based algorithm that maintains the set of all MAWs for a sliding window in $O(\sigma n)$ time using $O(\sigma d)$ working space. Crochemore et al. [9] also showed how their algorithm can be applied to approximate pattern matching under the length weighted index (LWI) metric [6].

The (in)efficiency of their algorithms is heavily dependent on combinatorial properties of MAWs for the sliding window. In particular, Crochemore et al. [9] studied the number of MAWs to be added/deleted when the current window is shifted to the right by one character. As was done in [9], for ease of discussion let us separately consider

- adding a new character T[i+d] to the current window T[i..i+d-1] of length d which forms T[i..i+d], and
- deleting the left-most character T[i-1] from the current window T[i-1..i+d-1] which forms T[i..i+d-1] of length d.

We remark that these two operations are symmetric.

Crochemore et al. [9] considered how many MAWs can change before and after the window has been shifted by one position, and showed that

$$|\mathsf{MAW}(T[i..i+d]) \triangle \mathsf{MAW}(T[i..i+d-1])| \le (s_i - s_\alpha)(\sigma - 1) + \sigma + 1, \\ |\mathsf{MAW}(T[i-1..i+d-1]) \triangle \mathsf{MAW}(T[i..i+d-1])| \le (p_i - p_\beta)(\sigma - 1) + \sigma + 1,$$

where \triangle denotes the symmetric difference and

• s_i is the length of the longest repeating suffix of T[i..i + d - 1],

¹The original claimed bound in [2] is O(n), however, the authors assumed that $\sigma = O(1)$.

- s_{α} is that of the longest suffix of T[i..i+d-1] having an internal occurrence immediately followed by $\alpha = T[i+d]$,
- p_i is that of the longest repeating prefix of T[i..i+d-1], and
- p_{β} is that of the longest prefix of T[i..i+d-1] having an internal occurrence immediately preceded by $\beta = T[i-1]$.

Since both $s_i - s_\alpha$ and $p_i - p_\beta$ can be at most d-1 in the worst case, the asymptotic bounds for the numbers of changes in the set of MAWs obtained by Crochemore et al. [9] are:

$$|\mathsf{MAW}(T[i..i+d]) \triangle \mathsf{MAW}(T[i..i+d-1])| \in O(\sigma d), \\ |\mathsf{MAW}(T[i-1..i+d-1]) \triangle \mathsf{MAW}(T[i..i+d-1])| \in O(\sigma d).$$
 (1)

Crochemore et al. [9] also considered the *total changes* in the set of MAWs for every sliding window over the string T, and showed that

$$\sum_{i=1}^{n-d} \left(\left| \mathsf{MAW}(T[i..i+d-1]) \bigtriangleup \mathsf{MAW}(T[i+1..i+d]) \right| \right) \in O(\sigma n). \tag{2}$$

1.3 Our contribution

The goal of this paper is to give more rigorous analyses on the number of MAWs for the sliding window model. This study is well motivated since revealing more combinatorial insights to the sets of MAWs for the sliding windows can lead to more efficient algorithms for computing them.

In this paper, we first give the following upper bounds:

$$|\mathsf{MAW}(T[i..i+d]) \triangle \mathsf{MAW}(T[i..i+d-1])| \leq d+\sigma'+1, \\ |\mathsf{MAW}(T[i-1..i+d-1]) \triangle \mathsf{MAW}(T[i..i+d-1])| \leq d+\sigma'+1,$$
 (3)

where σ' is the number of distinct characters in T[i..i+d-1]. We then show that our new upper bounds in (3) are tight by showing a family of strings achieving these bounds.

Since $\sigma' \leq d$ always holds, we immediately obtain new asymptotic upper bounds

$$|\mathsf{MAW}(T[i..i+d]) \triangle \mathsf{MAW}(T[i..i+d-1])| \in O(d), \\ |\mathsf{MAW}(T[i-1..i+d-1]) \triangle \mathsf{MAW}(T[i..i+d-1])| \in O(d).$$

Our new upper bounds in (4) improve Crochemore et al.'s upper bounds in (1) for any alphabet of size $\sigma \in \omega(1)$. Our upper bounds in (4) are also *tight* as there exists a family of strings achieving the matching lower bounds $\Omega(d)$.

In this paper, we also present a new upper bound for the total changes of MAWs:

$$\sum_{i=1}^{n-d} \left(|\mathsf{MAW}(T[i..i+d-1]) \triangle \mathsf{MAW}(T[i+1..i+d])| \right) \in O(\min\{\sigma, d\}n) \tag{5}$$

which improves the previous bound $O(\sigma n)$ in (2). We then show that this new upper bound in (5) is also tight.

All of our new bounds afore-mentioned are tight for any alphabet of size $\sigma' \geq 3$. We further explore the case of binary alphabets with $\sigma' = 2$, and show that there exist even tighter bounds in the binary case. Namely, for $\sigma' = 2$, we prove that

$$|\mathsf{MAW}(T[i..i+d]) \triangle \; \mathsf{MAW}(T[i..i+d-1])| \; \leq \; \max\{3,d\}, \\ |\mathsf{MAW}(T[i-1..i+d-1]) \triangle \; \mathsf{MAW}(T[i..i+d-1])| \; \leq \; \max\{3,d\}.$$

We remark that plugging $\sigma' = 2$ into (3) for the general case only gives $d + \sigma' + 1 = d + 3$, which is larger than $\max\{3, d\}$ in (6). We also show that the upper bounds $\max\{3, d\}$ in (6) are *tight* by giving the matching lower bounds with a family of binary strings.

A part of the results reported in this article appeared in a preliminary version of this paper [15].

2 Preliminaries

2.1 Strings

Let Σ be an alphabet. An element of Σ is called a character. An element of Σ^* is called a string. The length of a string T is denoted by |T|. The empty string ε is the string of length 0. If T = xyz, then x, y, and z are called a *prefix*, substring, and suffix of T, respectively. They are called a *proper prefix*, *proper substring*, and *proper suffix* of T if $x \neq T$, $y \neq T$, and $z \neq T$, respectively.

For any $1 \le i \le |T|$, the *i*-th character of T is denoted by T[i]. For any $1 \le i \le j \le |T|$, T[i..j] denotes the substring of T starting at i and ending at j. For convenience, $T[i'..j'] = \varepsilon$ for i' > j'. For any $1 \le i \le |T|$, let T[..i] = T[1..i] and T[i..] = T[i..|T|].

We say that a string w occurs in a string T if w is a substring of T. Note that by definition the empty string ε is a substring of any string T and hence ε always occurs in T.

2.2 Minimal absent words (MAWs)

A string w is called an absent word for a string T if w does not occur in S. An absent word w for S is called a minimal absent word or MAW for S if any proper substring of w occurs in S. We denote by MAW(S) the set of all MAWs for S. By the definition of MAWs, it is clear that $w \in MAW(S)$ iff the three following conditions hold:

- (A) w does not occur in S;
- (B) w[2...] occurs in S;
- (C) w[..|w|-1] occurs in S.

We note that if w is a string of length 1 which does not occur in S (i.e. w is a single character in Σ not occurring in S), then w is a MAW for T since $w[2..] = w[..|w| - 1] = \varepsilon$ is a substring of S.

2.3 MAWs for a sliding window

Given a string T of length n and a sliding window $S_i = T[i..j]$ of length d = j - i + 1 for increasing i = 1, ..., n - d + 1, our goal is to analyze how many MAWs for the sliding window can change when the window shifts over the string T. We will consider both the maximum change per one shift, and the maximum change for all the shifts over the input string.

As was done in [9], for simplicity, we separately consider two symmetric operations of appending a new character to the right of the window and of deleting the leftmost character from the window.

Example 1. Consider to append character c to the right of string cbaaaa. Then,

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 \begin{array}{lll} \mathsf{MAW}(\mathsf{cbaaaa}) & = & \{\mathsf{cc}, \mathsf{bb}, \mathsf{aaaaa}, \mathsf{bc}, \mathsf{ab}, \mathsf{ca}, \mathsf{ac}\}, \\ \mathsf{MAW}(\mathsf{cbaaaac}) & = & \{\mathsf{cc}, \mathsf{bb}, \mathsf{aaaaa}, \mathsf{bc}, \mathsf{ab}, \mathsf{ca}, \mathsf{acb}, \mathsf{bac}, \mathsf{baac}, \mathsf{baaac}\}. \\ \end{array}
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Thus $MAW(cbaaaa) \triangle MAW(cbaaaac) = \{\underline{ac}, acb, bac, baac, baaac\}$, where the underlined string is deleted from and the strings without underlines are added to the set of MAWs by appending c to cbaaaa.

3 Tight bounds on the changes to MAWs for sliding window

In this section, we present our new bounds for the changes of MAWs for the sliding window over the string T. In Section 3.1, we consider the number of changes of MAWs when the current window T[i..j] is extended by adding a new character T[j+1]. Section 3.2 is for the symmetric case where the leftmost character T[i] is deleted from T[i..i+j+1]. Finally, in Section 3.3, we consider the total number of changes of MAWs while the window has been shifted from the beginning of T until its end.

3.1 Changes to MAWs when appending character to right

We consider the number of changes of MAWs when appending T[j+1] to the current window T[i...j].

For the number of deleted MAWs, the next lemma is known:

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Lemma 1 ([9]). For any 1 \le i \le j < n, |MAW(T[i..j]) \setminus MAW(T[i..j+1])| = 1.
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Next, we consider the number of added MAWs. We classify each MAW w in $\mathsf{MAW}(T[i..j+1]) \setminus \mathsf{MAW}(T[i..j])$ to the following three types² (see Figure 1). A MAW w in $\mathsf{MAW}(T[i..j+1]) \setminus \mathsf{MAW}(T[i..j])$ is said to be of:

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Type 1 if neither w[2...] nor w[...|w|-1] occurs in T[i..j];
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Type 2 if w[2...] occurs in T[i...j] but w[...|w|-1] does not occur in T[i...j];

Type 3 if w[2..] does not occur in T[i..j] but w[..|w|-1] occurs in T[i..j].

We denote by \mathcal{M}_1 , \mathcal{M}_2 , and \mathcal{M}_3 the sets of MAWs of Type 1, Type 2 and Type 3, respectively. Recall that w is a MAW for T[i..j+1].

Let σ' be the number of distinct characters occurring in the current window T[i..j].

²At least one of w[2..] and w[..|w|-1] does not occur in T[i..j], since $w \notin \mathsf{MAW}(T[i..j])$.

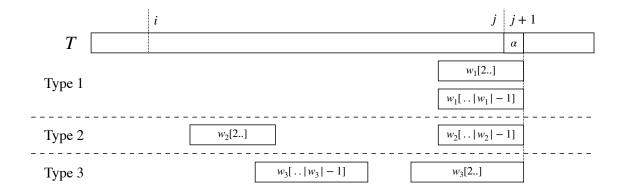


Figure 1: Illustration for the three types of MAWs, where $w_1 \in \mathcal{M}_1$, $w_2 \in \mathcal{M}_2$, and $w_3 \in \mathcal{M}_3$.

Lemma 2 ([9]). For any $1 \le i \le j < n$, $|\mathcal{M}_1| \le 1$. Also, if α is the character appended to T[i..j], then the only element of \mathcal{M}_1 is of form α^k with some $k \ge 1$.

Lemma 3. For any $1 \le i \le j < n$, $|\mathcal{M}_2| \le \sigma'$.

Proof. It is shown in [9] that the last characters of all MAWs in \mathcal{M}_2 are all distinct. Furthermore, by the definition of \mathcal{M}_2 , the last character T[j+1] of each MAW in \mathcal{M}_2 must occur in the current window T[i..j]. Thus, $|\mathcal{M}_2| \leq \sigma'$.

The next lemma holds for \mathcal{M}_3 :

Lemma 4. For any $1 \le i \le j < n$, $|\mathcal{M}_3| \le d - 1$, where d = j - i + 1.

Proof. We show that there is an injection $f: \mathcal{M}_3 \to [i, j-1]$ which maps each MAW $w \in \mathcal{M}_3$ to the ending position of the leftmost occurrence of w[..|w|-1] in the current window T[i..j].

First, we show that the range of this function f is [i, j-1]. By definition, w is absent from T[i..j+1] and w[|w|] = T[j+1] for each $w \in \mathcal{M}_3$, and thus, no occurrence of w[..|w|-1] in T[i..j] ends at position j. Hence, the range of f does not contain the position j, i.e. it is [i, j-1].

Next, for the sake of contradiction, we assume that f is not an injection, i.e. there are two distinct MAWs $w_1, w_2 \in \mathcal{M}_3$ such that $f(w_1) = f(w_2)$. W.l.o.g., assume $|w_1| \geq |w_2|$. Since $w_1[|w_1|] = w_2[|w_2|] = T[j+1]$ and $f(w_1) = f(w_2)$, w_2 is a suffix of w_1 . If $|w_1| = |w_2|$, then $w_1 = w_2$ and it contradicts with $w_1 \neq w_2$. If $|w_1| > |w_2|$, then w_2 is a proper suffix of w_1 , and it contradicts with the fact that w_2 is absent from T[i..j+1] (see Figure 2). Therefore, f is an injection and $|\mathcal{M}_3| \leq j-1-i+1=d-1$.

Summing up all the upper bounds for \mathcal{M}_1 , \mathcal{M}_2 , and \mathcal{M}_3 , we obtain the following:

Lemma 5. For any $1 \le i \le j < n$, $|\mathsf{MAW}(T[i..j+1]) \setminus \mathsf{MAW}(T[i..j])| \le \sigma' + d$, where d = j - i + 1.

Proof. Immediately follows from Lemmas 2, 3, and 4 and that \mathcal{M}_1 , \mathcal{M}_2 , and \mathcal{M}_3 are mutually disjoint.

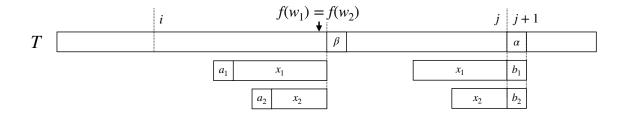


Figure 2: Illustration for the contradiction in the proof of Lemma 5. Consider two strings $w_1 = a_1x_1b_1$ and $w_2 = a_2x_2b_2$ that are MAWs for T of Type 3 where $a_1, a_2, b_1, b_2 \in \Sigma$ and $x_1, x_2 \in \Sigma^*$. If $|w_1| > |w_2|$ and $f(w_1) = f(w_2)$, then x_2 is a proper suffix of x_1 , and it contradicts that $a_2x_2b_2$ is absent from T.

Now we obtain the main result of this subsection, which shows the matching upper and lower bounds for $|\mathsf{MAW}(T[i..j+1]) \triangle \mathsf{MAW}(T[i..j])|$.

Theorem 1. For any $1 \le i \le j < n$, $|\mathsf{MAW}(T[i..j+1]) \triangle \mathsf{MAW}(T[i..j])| \le \sigma' + d + 1$, where d = j - i + 1. The upper bound is tight when $\sigma \ge 3$ and $\sigma' + 1 \le \sigma$.

Proof. By Lemma 1 and Lemma 5, we have $|\mathsf{MAW}(T[i..j+1]) \triangle \mathsf{MAW}(T[i..j])| = |\mathsf{MAW}(T[i..j+1]) \setminus \mathsf{MAW}(T[i..j])| + |\mathsf{MAW}(T[i..j]) \setminus \mathsf{MAW}(T[i..j+1])| \leq \sigma' + d + 1.$

In the following, we show that the upper bound is tight, i.e. there is a string Z of length d and a character α where $|\mathsf{MAW}(Z) \triangle \mathsf{MAW}(Z\alpha)| = \sigma' + d + 1$ for any two integers d and σ' with $1 \le \sigma' \le d$ and $\sigma' + 1 \le \sigma$. Let $\Sigma = \{a_1, a_2, \cdots, a_\sigma\}$ be an alphabet. Given two integers d and σ' with $1 \le \sigma' \le d$ and $\sigma' + 1 \le \sigma$, consider a string $Z = a_1 a_2 \cdots a_{\sigma'-1} a_{\sigma'}^{d-\sigma'+1}$ of length d and a character $\alpha = a_{\sigma'+1}$. Then,

$$MAW(Z) \setminus MAW(Z\alpha) = \{\alpha\}.$$

Also,

$$\mathsf{MAW}(Z\alpha) \setminus \mathsf{MAW}(Z) = \{\alpha^2\} \cup \{\alpha a_i \mid 1 \le i \le \sigma'\} \cup \{a_i \alpha \mid 1 \le i \le \sigma' - 1\}$$
$$\cup \{a_{\sigma'-1} a_{\sigma'}^e \alpha \mid 1 \le e \le d - \sigma'\}.$$

This leads to the matching lower bound $|\mathsf{MAW}(Z) \triangle \mathsf{MAW}(Z\alpha)| = \sigma' + d + 1$.

A concrete example for our lower-bound strings Z and $Z\alpha$ is shown below.

Example 2. Let Z = abcddd where d = |Z| = 6, $\sigma' = 4$, and $d - \sigma' + 1 = 3$. Also, let $\alpha = e$. Then,

$$MAW(abcddd) \setminus MAW(abcddde) = \{e\}$$

and

$$\begin{split} \mathsf{MAW}(\mathsf{abcddde}) \setminus \mathsf{MAW}(\mathsf{abcddd}) \\ &= \ \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \\ &= \ \{\mathsf{ee}\} \cup \{\mathsf{ea}, \mathsf{eb}, \mathsf{ec}, \mathsf{ed}\} \cup \{\mathsf{ae}, \mathsf{be}, \mathsf{ce}, \mathsf{cde}, \mathsf{cdde}\}, \end{split}$$

and therefore $|\mathsf{MAW}(Z) \triangle \mathsf{MAW}(Z\alpha)| = \sigma' + d + 1 = 11$.

3.2 Changes to MAWs when deleting leftmost character

Next, we analyze the number of changes of MAWs when deleting the leftmost character from a string. By a symmetric argument to Theorem 1, we obtain:

Corollary 1. For any $1 < i \le j \le n$, $|\mathsf{MAW}(T[i..j]) \triangle \mathsf{MAW}(T[i-1..j])| \le \sigma' + d + 1$ where d = j - i + 1 and σ' is the number of distinct characters occurs in T[i..j]. Also, the upper bound is tight when $\sigma \ge 3$ and $\sigma' + 1 \le \sigma$.

Proof. Symmetric to the proof of Lemma 1.

Finally, by combining Theorem 1 and Corollary 1, we obtain the next theorem:

Theorem 2. Let d be the window length. For any string T of length n > d and each position i in T with $1 \le i \le n-d$, $|\mathsf{MAW}(T[i..i+d-1]) \triangle \mathsf{MAW}(T[i+1..i+d])| \in O(d)$. Also, there exists a string T' with $|T'| \ge d+1$ which satisfies $|\mathsf{MAW}(T'[j..j+d-1]) \triangle \mathsf{MAW}(T'[j+1..j+d])| \in \Omega(d)$ for some j with $1 \le j \le |T'| - d$.

This theorem improves Crochemore et al.'s upper bound for $|\mathsf{MAW}(T[i..i+d-1]) \triangle \mathsf{MAW}(T[i+1..i+d])| \in O(\sigma d)$ for any alphabet of size $\sigma \in \omega(1)$.

3.3 Total changes of MAWs when sliding window on string

In this subsection, we consider the total number of changes of MAWs when sliding the window of length d from the beginning of T to the end of T. We denote the total number of changes of MAWs by $S(T,d) = \sum_{i=1}^{n-d} |\mathsf{MAW}(T[i..i+d-1]) \triangle \mathsf{MAW}(T[i+1..i+d])|$. The following lemma is known:

Lemma 6 ([9]). For a string T of length n > d over an alphabet Σ of size σ , $\mathcal{S}(T, d) \in O(\sigma n)$.

The aim of this subsection is to give a more rigorous bound for S(T, d). We first show that the above bound is tight under some conditions.

Lemma 7. The upper bound of Lemma 6 is tight when $\sigma \leq d$ and $n-d \in \Omega(n)$

Proof. If $\sigma = 2$, the lower bound $S(T', d) \in \Omega(n - d) = \Omega(\sigma(n - d))$ is obtained by string $T' = (ab)^{n/2}$.

In the sequel, we consider the case where $\sigma \geq 3$. Let k be the integer with $(k-1)(\sigma-1) \leq d < k(\sigma-1)$. Note that $k \geq 2$ since $\sigma \leq d$. Let $\Sigma = \{a_1, a_2, \cdots, a_\sigma\}$ and $\alpha = a_\sigma$. We consider a string $T' = U^e + U[..m]$ where $U = a_1\alpha^{k-1}a_2\alpha^{k-1} \dots a_{\sigma-1}\alpha^{k-1}$, $e = \lfloor \frac{n}{k(\sigma-1)} \rfloor$, and $m = n \mod k(\sigma-1)$. Let c be a character that is not equal to α . For any two distinct occurrences $i_1, i_2 \in occ_{T'}(c)$ for $c, |i_1 - i_2| \geq k(\sigma-1) > d$. Thus, any character $c \neq \alpha$ is absent from at least one of two adjacent windows T'[i..i + d - 1] and T'[i + 1..i + d] for every $1 \leq i \leq n - d$.

Now we consider a window W = T[p-d..p-1] where $d+1 \leq p \leq n$ and $T[p] = \beta \neq \alpha$. Let $\Pi = \{b_1, b_2, \cdots, b_\pi, \alpha\} \subset \Sigma \setminus \{\beta\}$ be a set of all characters that occur in W. W.l.o.g., we assume that the current window is $W = \alpha^r b_1 \alpha^{k-1} b_2 \alpha^{k-1} \cdots b_\pi \alpha^{k-1}$ and the next window is $W' = W[2..]\beta$ where $r = d \mod k$ (see Figure 3). For any character $b \in \Pi \setminus \{b_1, b_\pi, \alpha\}$, $b\alpha^\ell\beta$ is in $\mathsf{MAW}(W') \triangle \mathsf{MAW}(W)$ for every $0 \leq \ell \leq k-1$. If r > 0, $b_1\alpha^\ell\beta$ is also in $\mathsf{MAW}(W') \triangle \mathsf{MAW}(W)$ for every $0 \leq \ell \leq k-1$. Otherwise, b_1 is in $\mathsf{MAW}(W') \triangle \mathsf{MAW}(W)$

$$\Sigma = \{a, b, c, d\}, d = 9$$

$$W = T[4..12] \qquad W' = T[5..13]$$

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17$$

$$T = b \quad a \quad a \quad c \quad a \quad a \quad d \quad a \quad a \quad b \quad a \quad a \quad c$$

Figure 3: Illustration of examples of MAWs for adjacent two windows. In this example, $\sigma = 4, d = 9$, and k = 4. The size of the symmetric difference of $\mathsf{MAW}(W)$ and $\mathsf{MAW}(W')$ is $|\mathsf{MAW}(W) \triangle \mathsf{MAW}(W')| = |\{\mathsf{b}, \mathsf{aac}, \mathsf{cac}, \mathsf{dac}, \mathsf{ac}, \mathsf{ba}, \mathsf{bb}, \mathsf{bc}, \mathsf{bd}, \mathsf{cb}, \mathsf{cab}, \mathsf{caab}, \mathsf{caaab}, \mathsf{dab}, \mathsf{dab}\}| = 16.$

and $b_1\alpha^\ell b_2$ is in MAW(W') \triangle MAW(W) for every $0 \le \ell \le k-2$ since b_1 is absent from W'. Also, β is in MAW(W') \triangle MAW(W) and $b_\pi\alpha^\ell\beta$ is in MAW(W') \triangle MAW(W) for every $0 \le \ell \le k-2$. Thus, at least $(\pi-2)k+k+1+(k-1)=\pi k$ MAWs are in MAW(W') \triangle MAW(W). Additionally, the number π of distinct characters which occur in W and are not equal to α is at least $\lfloor (\sigma-1)/2 \rfloor$, since $k\lfloor (\sigma-1)/2 \rfloor \le k(\sigma-1)/2 = (k-k/2)(\sigma-1) \le (k-1)(\sigma-1) \le d$. Therefore, $|\mathsf{MAW}(W') \triangle \mathsf{MAW}(W)| \ge \pi k \ge \lfloor (\sigma-1)/2 \rfloor k \in \Omega(\sigma k) = \Omega(d)$. The number of pairs of two adjacent windows W and W' where $|\mathsf{MAW}(W') \triangle \mathsf{MAW}(W)| \in \Omega(d)$ is $\Theta((n-d)/k)$. Therefore, we obtain $S(T',d) \in \Omega(d(n-d)/k) = \Omega(\sigma(n-d)) = \Omega(\sigma n)$ since $n-d \in \Omega(n)$.

Next, we consider the case where $\sigma \geq d+1$.

Lemma 8. For a string T of length n > d over an alphabet Σ of size σ , $S(T, d) \in O(d(n-d))$, and this upper bound is tight when $\sigma \geq d+1$

Proof. By Corollary 2, it is clear that $\mathcal{S}(T,d) \in O(d(n-d))$. Next, we show that there is a string T' of length n > d such that $\mathcal{S}(T',d) \in \Omega(d(n-d))$ for any integer d with $1 \le d \le \sigma - 1$. Let $\Sigma = \{a_1, a_2, \cdots, a_\sigma\}$. We consider a string $T' = (a_1 a_2 \cdots a_{d+1})^e a_1 a_2 \cdots a_k$ where $e = \lfloor n/(d+1) \rfloor$ and $k = n \mod (d+1)$. For each window W = T'[i..i+d-1] in T', W consists of distinct d characters, and the character T'[i+d] that is the right neighbor of W is different from any of characters occur in W. W.l.o.g., we assume that the current window is $W = a_1 a_2 \cdots a_d$ and the next window is $W' = W[2..]a_{d+1}$. Then, $|\mathsf{MAW}(W') \triangle \mathsf{MAW}(W)| = |\{a_{d+1}\}\} \cup \{a_{d+1}a_i \mid 2 \le i \le d\}| = d$. Therefore, $\mathcal{S}(T',d) = d(n-d)$.

The main result of this section follows from the above lemmas:

Theorem 3. For a string T of length n > d over an alphabet Σ of size σ , $S(T,d) \in O(\min\{d,\sigma\}n)$. This upper bound is tight when $n-d \in \Omega(n)$.

We remark that $n - d \in \Omega(n)$ covers most interesting cases for the window length d, since the value of d can range from O(1) to cn for any 0 < c < 1.

4 Tighter bounds for binary alphabets

In this section we consider the case where $\sigma' = 2$, i.e. when both the current sliding window S = T[i..i + d - 1] and the next window $S\alpha = T[i..i + d]$ extended with a new character $\alpha = T[i+d]$ consist of two distinct characters. The goal of this section is to show that when $\sigma' = 2$, there exists a tighter upper bound for the number of changes of MAWs than the general case with $\sigma' \geq 3$. In what follows, let us denote by $\Sigma_2 = \{0, 1\}$ the binary alphabet, and assume w.l.o.g. that we append the new character $\alpha = 0$ to the window S of length d and obtain the extended window $S\alpha = S0$.

As a warm up, we begin with the two following lemmas which show that at most 3 MAWs can change in the cases where d = 1 and d = 2 for any binary strings.

Lemma 9. For any string S0 over Σ_2 with |S| = d = 1, $|\mathsf{MAW}(S) \triangle \mathsf{MAW}(S0)| \leq 3$.

Proof. For each $S \in \{0, 1\}$,

```
MAW(0) \triangle MAW(00) = \{\underline{00}, 000\},\

MAW(1) \triangle MAW(10) = \{\underline{0}, 00, 01\},\
```

where the underlined strings are those in $MAW(S) \setminus MAW(S0)$ and the strings without underlines are those in $MAW(S0) \setminus MAW(S)$. Thus the lemma holds.

Lemma 10. For any string S0 over Σ_2 with |S| = d = 2, $|\mathsf{MAW}(S) \triangle \mathsf{MAW}(S0)| \leq 3$.

Proof. For each $S \in \{00, 01, 10, 11\}$,

```
\begin{array}{llll} \mathsf{MAW}(00) \bigtriangleup \mathsf{MAW}(000) & = & \{\underline{000},0000\}, \\ \mathsf{MAW}(01) \bigtriangleup \mathsf{MAW}(010) & = & \{\underline{10},101\}, \\ \mathsf{MAW}(10) \bigtriangleup \mathsf{MAW}(100) & = & \{\underline{00},000\}, \\ \mathsf{MAW}(11) \bigtriangleup \mathsf{MAW}(110) & = & \{0,00,01\}, \end{array}
```

where the underlined strings are those in $MAW(S) \setminus MAW(S0)$ and the strings without underlines are those in $MAW(S0) \setminus MAW(S)$. Thus the lemma holds.

We move onto the case where $d \geq 3$. Our first observation is that we can restrict ourselves to the case where S is not unary. For any d, it is clear that $|\mathsf{MAW}(0^d) \triangle \mathsf{MAW}(0^{d+1})| = 2$. Now let us consider 1^d in the next lemma.

Lemma 11. For any $d \geq 3$ let $V = 1^d$. Then, there exists another string S of length d over Σ_2 such that S[k] = 0 for some $1 \leq k \leq d$ and $|\mathsf{MAW}(V) \triangle \mathsf{MAW}(V0)| \leq |\mathsf{MAW}(S) \triangle \mathsf{MAW}(S0)|$.

Proof. Since $V = 1^d$, $\mathsf{MAW}(V) \setminus \mathsf{MAW}(V0) = \{0\}$. Also, $\mathsf{MAW}(V0) \setminus \mathsf{MAW}(V) = \{00, 01\}$. Thus $|\mathsf{MAW}(V) \triangle \mathsf{MAW}(V0)| = 3$ for any $d \ge 1$.

Let $S=01^{d-1}$ and $S0=01^{d-1}0$ with $d\geq 3$. Then, $\mathsf{MAW}(S0)\setminus \mathsf{MAW}(S)=\{01^k0\mid 1\leq k\leq d-2\}\cup\{101\}$ and $\mathsf{MAW}(S0)\setminus \mathsf{MAW}(S)=\{10\}$. Thus we have $|\mathsf{MAW}(S)\triangle \mathsf{MAW}(S0)|\geq d\geq 3$.

According to Lemmas 9, 10 and 11, in what follows we focus on the case where $d \ge 3$ and the current window S = T[i..i+d-1] contains at least a 0. The latter condition implies that we focus on the case where the new character $\alpha = 0$ already occurs in the window S.

As in the case of non-binary alphabets, we analyze the numbers of added Type-1/Type-2/Type-3 MAWs in $\mathcal{M}_1/\mathcal{M}_2/\mathcal{M}_3$ for binary strings. Recall that in the current context, a MAW w for S0 = T[i..i+d] is said to be of:

- Type 1 if neither w[2..] nor w[..|w|-1] occurs in S;
- Type 2 if w[2..] occurs in S but w[..|w|-1] does not occur in S;
- Type 3 if w[2..] is does not occur in S but w[..|w|-1] occurs in S.

We first show the upper bound for the size of \mathcal{M}_3 in the case where $\sigma' = 2$.

Lemma 12. For any binary string S0 over Σ_2 such that $|S| = d \ge 3$, $|\mathcal{M}_3| \le d - 2$.

Proof. Recall the proof for Lemma 4. There, we proved that each MAW w of Type 3 for any non-binary string $R\alpha = T[i..i+d] = T[i..j+1]$ is mapped by an injection f to a distinct position of T[i..j] in range [i,j-1], or alternatively to a distinct position of R in range [1,d-1]. This showed $|\mathcal{M}_3| \leq d-1$ for $\sigma' \geq 3$.

Here we show that the range of such an injection f is [2, d-1] for any binary string S with $\sigma' = 2$. Since the appended character is $\alpha = 0$, and since the candidate x for the MAW of Type 3 which should be mapped to the first position in S is of length 2, the candidate x has to be either 00 or 10.

- (1) If x = 00, then S[1] = 0. If 00 does not occur in S (see also the top picture of Figure 4), then 00 is already a MAW for S (i.e. $00 \in \mathsf{MAW}(S)$). Thus $00 \notin \mathsf{MAW}(S) \setminus \mathsf{MAW}(S)$ in this case. Otherwise (00 occurs in S), then clearly 00 is not a MAW for S0 (see also the middle picture of Figure 4).
- (2) If x = 10, then S[1] = 1. However, since the appended character is 0, 10 must occur somewhere in S0 (see also the bottom picture of Figure 4). Thus 10 is not a MAW for S0.

Hence, the first position of S cannot be assigned to any MAW of Type 3 for S0, leading to $|\mathcal{M}_3| \leq d-2$ for any binary string S of length $d \geq 3$.

In other words, Lemma 12 shows that in the binary case with $\sigma' = 2$, the maximum number of added Type-3 MAWs is 1 less than in the case with $\sigma' = 3$.

Next, we consider the total number of added Type-1/Type-2 MAWs.

Lemma 13. For any binary string S0 over Σ_2 such that $|S| = d \ge 3$ and S[i] = 0 for some $1 \le i \le d$, $|\mathcal{M}_1| + |\mathcal{M}_2| \le 2$.

Proof. Let k denote the length of the maximum run of 0's that is a suffix of S. If S[|S|] = 1 then let k = 0. By the definition of \mathcal{M}_1 , 0^{k+2} is the only candidate for a Type-1 MAW for S0, in which case $au = ub = 0^{k+1}$ occurs only once in S0 as a suffix of S0. This means that 0^{k+2} can be a Type-1 MAW for S0 only if 0^k is the longest run of 0's in S.

Now suppose that 0^{k+2} is a Type-1 MAW for S0, and let a'u'0 denote a Type-2 MAW for S0. Then, by definition, u'0 is a suffix of S0 (see also the middle of Figure 1).

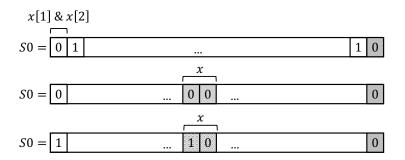


Figure 4: Characteristics of Type-3 MAWs in the binary case with $\sigma' = 2$, where the rightmost 0 in gray is the new appended character in each picture.

- If |u'| > k, then 0^{k+1} is a suffix of u' as shown in Figure 5. However, by the definition of Type-2 MAWs, au' must occur in S (see also the middle of Figure 1), which implies that 0^{k+1} occurs in S. This contradicts that 0^k is the longest run of 0's in S.
- If |u'| < k, then $a'u'0 = 0^p$ with p < k + 2 must be a suffix of S0, but this contradicts that a'u'0 is a MAW for S0.

Hence the only possible case where both Type-1 and Type-2 MAWs exist is |u'| = k. Thus $|\mathcal{M}_1| + |\mathcal{M}_2| \le 2$ for any binary string S0 such that S contains a 0.

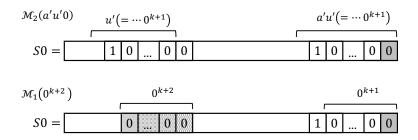


Figure 5: Collision between the new Type-2 MAW and Type-1 MAW in the binary case, where the rightmost 0 in gray is the new appended character in each picture.

A direct consequence of Lemma 12 and Lemma 13 is an upper bound for the added MAWs $|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| \le d$ for any binary string S0 with $|S| = d \ge 3$. In what follows, we further reduce this upper bound to $|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| \le d - 1$. For this purpose, we introduce the next lemma:

Lemma 14. For any binary string S0 over Σ_2 such that $|S| = d \ge 3$, $|\mathcal{M}_2|$ is at most the number of 0's in S[1..d-1], and $|\mathcal{M}_3|$ is at most the number of 1's in S[3..d].

Proof. Let aub denote a MAW in \mathcal{M}_2 , where $a, b \in \Sigma_2$ and $u \in \Sigma_2^*$.

First we consider Type-2 MAWs for S0. By the definition of \mathcal{M}_2 , au does not occur in S but au occurs in S0, which implies that the last character of au is 0. Consider two distinct Type-2 MAWs aub and a'u'b where $a, a', b \in \Sigma_2$, $u, u' \in \Sigma_2^*$, and |aub| < |a'u'b|. Since both aub and a'u'b are Type-2 MAWs for S0, both S00, both S00 are suffixed of S01. In addition,

au is a proper suffix of a'u' since |aub| < |a'u'b|. This implies that ub and u'b cannot have occurrences in S with the same ending positions, since otherwise aub must occur in S, a contradiction. Since the last characters of au and a'u' are both 0, they cannot share the same ending positions in S. In addition, the last position d = |S| in S cannot be the ending position of au for any Type-2 MAW aub since aub does not occur in S0. Thus, the total number of Type-2 MAWs for S0 is upper bounded by the number of 0's in S[1...d-1].

Second we consider Type-3 MAWs for S0. Since ub is a suffix of S0, each Type-3 MAW is of form au0. Also, by the definition of \mathcal{M}_3 , there has to be an occurrence of au in S. Note that this occurrence has to be immediately followed by a 1 since otherwise au0 must occur in S0, a contradiction. Thus, for each Type-3 MAW au0 of S0, we need an occurrence of au1 in S. Since $|au| \geq 1$, we clearly cannot use the first position of S as the ending position of S and the ending position of S and the ending position of S cannot be the ending position of S and S are the ending position of S are the ending position of S and S are the ending position of S and S are the ending position of S and S are the ending position of S are the ending position of S and S are the ending position of S and S are the ending position of S and S are the ending position of S and S are the ending position of S and S are the ending position of S are the ending position of S are the ending position of S are the e

Intuitively, Lemma 14 implies that flipping 0's and/or 1's in S[3..d-1] does not increase the total number of Type-2 and Type-3 MAWs for S0.

Lemma 15. For any binary string S0 over Σ_2 with $|S| = d \ge 3$, $|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| \le d - 1$.

Proof. Let U be a binary string over Σ_2 such that U0 has the maximum total number of added MAWs. It follows from Lemma 12 and Lemma 14 that $U[3..d] = 1^{d-2}$. Recall that by Lemma 2 Type-1 MAW must be of form 0^h and 0^{h-1} has to be a suffix of U. However, since (U0)[d..d+1] = 10 and U contains at least a 0 by Lemma 11, the only candidate 0 cannot be a Type-1 MAW for U0. Thus there is no Type-1 MAW for U0. Using Lemma 13, we can now conclude that U[1..2] = 00, and thus $U = 001^{d-2}$. Now the sets of all the added MAWs for $U0 = 001^{d-2}0$ are

$$\mathcal{M}_1 = \emptyset,$$

 $\mathcal{M}_2 = \{100, 101\},$
 $\mathcal{M}_3 = \{01^k0 \mid 1 \le k \le d - 3\},$

which leads to $|\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = d - 1$ for U0. Since any other string V0 with |V| = d has less added MAWs than U0, the lemma holds.

Our main theorem immediately follows from Lemma 1 and Lemma 15:

Theorem 4. For any binary string S over Σ_2 with $|S| = d \ge 3$, $|\mathsf{MAW}(S) \triangle \mathsf{MAW}(S0)| \le d$, and this upper bound is tight.

The next corollary, which immediately follows Lemmas 9, 10, and Theorem 4, summarizes the results of this section.

Corollary 2. For any binary string S over Σ_2 with |S| = d, $|\mathsf{MAW}(S) \triangle \mathsf{MAW}(S0)| \le \max\{3,d\}$, and this upper bound is tight for any $d \ge 1$.

5 Conclusions and future work

In this paper, we revisited the problem of computing the minimal absent words (MAWs) for the sliding window model, which was first considered by Crochemore et al. [9].

We investigated combinatorial properties of MAWs for a sliding window of fixed length d over a string of length n. Our contributions are matching upper and lower bounds for the number of changes in the set of MAWs for a sliding window when the window is shifted to the right by one character. For the general case where the window S and the extended window $S\alpha$ contain three or more distinct characters (i.e. $\sigma' \geq 3$), the number of changes in the set of MAWs for S and $S\alpha$ is at most $d + \sigma' + 1$ and this bound is tight. For the case of binary alphabets (i.e. $\sigma' = 2$), it is upper bounded by $\max\{3, d\}$ and this bound is also tight.

We also gave an asymptotically tight bound $O(\min\{d, \sigma\}n)$ for the number $\mathcal{S}(T, d)$ of total changes in the set of MAWs for every sliding window of length d over any string T of length n, where σ is the alphabet size for the whole input string T.

The following open questions are intriguing:

- We showed that a matching lower bound $S(T,d) \in \Omega(\min\{d,\sigma\}n)$ when $n-d \in \Omega(n)$. Is there a similar lower bound when $n-d \in o(n)$?
- Crochemore et al. [9] gave an online algorithm that maintains the set of MAWs for a sliding window of length d in $O(\sigma n)$ time. Can one improve the running time to optimal $O(\min\{d,\sigma\}n)$?

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